Homologcal algebra exercise sheet Week 2

1. Prove the following statements in an abelian category without using the Freyd-Mitchell embedding theorem:
   (a) Assume $0 \to A$ is a kernel for $f : A \to B$. Then f is monic. Dually if $B \to 0$ is a cokernel of f, then f is epi.
   (b) A morphism $f : A \to B$ that is both monic and epi is an isomorphism.
   (c) For any morphism $f : A \to B$ the induced morphism $A \to \mathrm{im}(f)$ is epi.

2. Given a category I and an abelian cateory $\mathcal{A}$, show that the functor category $\mathcal{A}^I$ is also an abelian category and that the kernel of $\eta : B \to C$ is the functor $i \mapsto \ker(\eta(i))$.

3. Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors. That is, there is a natural isomorphism

$$\tau : \mathrm{Hom}_{\mathcal{A}}(L(A), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

   Then L is right exact, and R is left exact.

4. (5-Lemma) Consider the following commutative diagram in an abelian category $\mathcal{A}$:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E. \end{tikzcd}$$

   Show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are isomorphisms, then so is c.

   Hint: You may use the Freyd-Mitchell embedding theorem